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AN OPTIMAL SHOCK–EXPANSION SYSTEM IN A STEADY GAS FLOW

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We study the steady, supersonic, uniform flow of an inviscid perfect gas passing successively through a simple Prandtl–Mayer expansion r and a shock j which have one direction. The system of two waves $S_2 = \{r, j\}$ is specified by two parameters: the Mach number M and the net angle of freestream turning β_s. As the systems of waves considered in [1–3], for certain strengths of the waves included in the system $S_2^{(f)}$, this system is optimal for most gas-dynamic variables f. Analytical solutions are presented which make it possible to determine monotonic and nonmonotonic ranges of gas-dynamic variables behind the system and to calculate the wave strengths when the system is optimal.

1. The system of waves $S_2 = \{r, j\}$ transforms the set of gas-dynamic variables $F = \{p, \rho, T, \rho v^2, p_0, \rho_0, T_0\}$ that characterize the free stream into the set $F_2 = \{p_2, \rho_2, T_2, \rho_2 v_2^2, p_{02}, \rho_{02}, T_{02}\}$ whose elements determine the flow properties behind the shock wave. The members f and f_2 of the sets F and F_2 are related by the following wave relations [1]:

$$I_s^{(f)} = \prod_{k=1}^{2} I_k^{(f)}, \qquad I_k^{(f)} = \frac{f_k}{f_{k-1}}.$$

Here f_{k-1} and f_k are variables ahead of and behind a wave ($f_{k-1} \equiv f$ for $k = 1$), the quantity $I_k^{(p)} \equiv J_k = p_k/p_{k-1}$ determines the strength of an individual wave, and the quantity $J_s = J_1 J_2$ determines the strength of the wave system.

Omel'chenko and Uskov proved [1] that for given values of M and the specific heat ratio γ, the gas-dynamic variables f_2 behind the wave system S_2 are expressed only in terms of the strength of the system J_s and the corresponding values of the variables f. In particular,

$$I_1^{(\rho)} \equiv E_1 = \rho_1/\rho = J_1^{1/\gamma}, \qquad I_2^{(\rho)} \equiv E_2 = \rho_2/\rho_1 = (J_2 + \varepsilon)/(1 + \varepsilon J_2), \qquad \varepsilon = (\gamma - 1)/(\gamma + 1),$$

$$I_s^{(T)} \equiv \Theta_s = T_2/T = J_s/E_s = \mu(\mathrm{M})/\mu(\mathrm{M}_2), \quad \mu(\mathrm{M}) = 1 + \varepsilon(\mathrm{M}^2 - 1),$$

$$I_s^{(d)} \equiv C_s = d_2/d = J_s(\mathrm{M}_2^2/\mathrm{M}^2), \quad d = \rho v^2.$$

The angle of rotation of flow in the system

$$\beta_s = \sum_{k=1}^{2} \psi_k \beta_k \qquad (\psi_1 = -1, \quad \psi_2 = +1) \tag{1.1}$$

is given by the relations

$$\beta_1^{(r)} = \omega(\mathrm{M}_1) - \omega(\mathrm{M}); \tag{1.2}$$

$$\beta_2^{(j)} = \arctan\left[\sqrt{\frac{J_m^{(1)} - J_2}{J_2 + \varepsilon}}\, \frac{(1 - \varepsilon)(J_2 - 1)}{J_m^{(1)} + \varepsilon - (1 - \varepsilon)(J_2 - 1)}\right], \tag{1.3}$$

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where the values of the Prandtl–Mayer function $\omega(\mathrm{M})$ are calculated from the Mach numbers ahead of the wave M and behind it M_1; the quantity $J_m^{(1)} = (1+\varepsilon)\mathrm{M}_1^2 - \varepsilon$ determines the strength of a normal shock in the flow with M_1. The functions (1.2) and (1.3) are analyzed by Uskov [4] in the plane of wave strengths β, $\Lambda = \ln J$.

We have previously shown [1] in a more general formulation for a system of n shocks that the quantities $I_s^{(f)}$ can have optimal values and determine an optimal system $S_n^{(f)}$ for an arbitrary gas-dynamic variable f. The quantities J_k in $S_n^{(f)}$ are found from the optimal conditions and determine the geometry of bodies for which $S_n^{(f)}$ is realized in the flow around these bodies. Such optimal systems are called gas-dynamically imposed.

Certain geometrical constraints are often imposed on aerodynamic bodies. For example, for supersonic flow past wedge–plate or cone–cylinder configurations, the net angle of flow turning β_s in the system of waves obtained is assumed to be given. Systems in which the strengths of their waves depend on geometrical constraints are called geometrically imposed. The numerical investigation of Grigorenko and Kraiko [5] of the system $S_2 = \{j, r\}$ obtained for flow past the bodies mentioned above with a given value of β_s indicates the nonmonotone behavior of variables f when M and β_s are varied. Hence, geometrically imposed systems can also be optimal.

In the present paper, we prove that for given values of M and β_s, the geometrically imposed system $S_2^{(f)} = \{r, j\}$ can be optimal for most of the variables f. The strengths $J_1^{(f)}$ and $J_2^{(f)}$ at which $S_2^{(f)}$ is realized depend on M and β_s and differ significantly from the corresponding strengths in optimal, gas-dynamically imposed systems.

Since the wave system $S_2 = \{r, j\}$ is often a supersonic-flow component, it is important to test it for an extremum in problems of aerodynamic design work. In particular, the data of the present paper, together with the results of [5], can serve as a basis for the design of optimal shapes of aircraft.

2. The optimal values of the functions $I^{(f)}(J_1, J_2)$ are found by the Lagrange multiplier method. For the chosen gas-dynamic variable f, the Lagrangian

$$L^{(f)} = I_s^{(f)} + \lambda\Big(\sum_{k=1}^{2}\psi_k\beta_k - \beta_s\Big) \tag{2.1}$$

depends on three variables: the wave strengths J_1 and J_2 and the Lagrange multiplier λ.

Differentiating (2.1) with respect to these variables, we obtain a system of two equations, one of which is the equation of constraint (1.1), and the other has the form

$$I_2^{(f)}\frac{\partial I_1^{(f)}}{\partial J_1}\,\frac{\partial(\psi_2\beta_2)}{\partial J_2} = I_1^{(f)}\frac{\partial I_2^{(f)}}{\partial J_2}\,\frac{\partial(\psi_1\beta_1+\psi_2\beta_2)}{\partial J_1}. \tag{2.2}$$

Using relations (1.1)–(1.3), we rewrite Eq. (2.2) as

$$I_2^{(f)}\frac{\partial I_1^{(f)}}{\partial J_1} + I_1^{(f)}\frac{\partial I_2^{(f)}}{\partial J_2}\left(\frac{\partial\beta_2}{\partial J_2}\right)^{-1}\left[\frac{\partial\omega(\mathrm{M}_1)}{\partial J_1} - \frac{\partial\beta_2}{\partial J_1}\right] = 0. \tag{2.3}$$

The derivatives in (2.3) are found by differentiating (1.2) and (1.3) with respect to the corresponding variables.

Relation (1.1) is a geometrical constraint on the domain of existence for the system S_2. For example, for $\beta_s > 0$, the inequality $\mathrm{M} > \mathrm{M}_*$ is a sufficient condition for the existence of the system, where the Mach number M_* corresponds to a shock that rotates the flow through the angle β_s and retards it to the speed of sound behind the shock ($\mathrm{M}_2 = 1$). The value of M_* is calculated from the formulas

$$\beta_s = \arctan\sqrt{\frac{J_*-1}{1+\varepsilon J_*}}\,\frac{(1-\varepsilon)(J_*-1)}{(J_*+\varepsilon)+(J_*-1)}, \quad J_* = \frac{\mu_*-1}{2\varepsilon} + \sqrt{\left(\frac{\mu_*-1}{2\varepsilon}\right)^2 + \mu_*}. \tag{2.4}$$

Figure 1a (curve 1) shows the function $\mathrm{M}_*(\beta_s)$ (the fragment marked by the dashed curve in Fig. 1a is given on an enlarged scale in Fig. 1b). Here and below, the calculation results are presented for $\gamma = 1.4$.

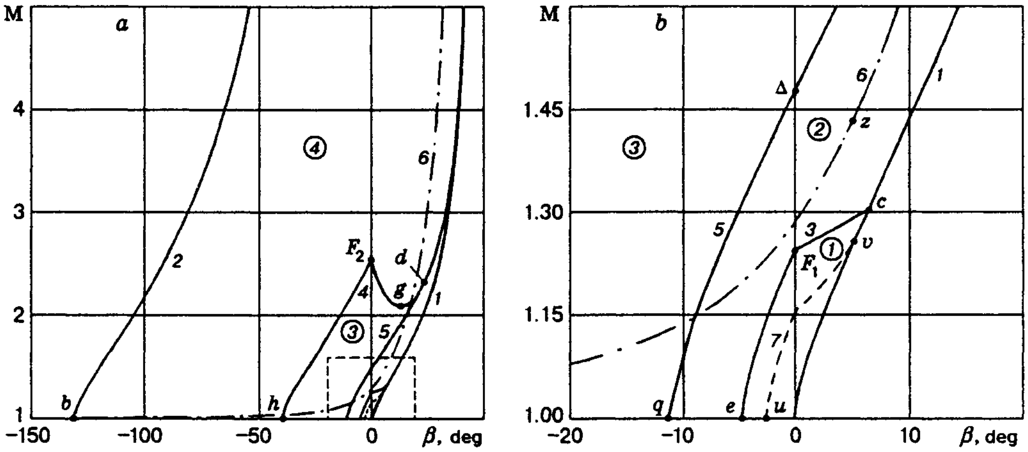

Fig. 1

For $M_* \to \infty$, the maximum angle of flow turning in S_2 is found from (2.4):

$$\beta_a = \arctan((1-\varepsilon)/(2\sqrt{\varepsilon})) \tag{2.5}$$

($\beta_a = 45.58°$)

Letting $M_1 \to \infty$ in (1.2), for $\beta_s < 0$ we determine the value of M_r that bounds from above the Mach number values ahead of an expansion wave capable of rotating the flow through the given angle β_s:

$$\frac{1}{\sqrt{\varepsilon}}\arctan\frac{1}{\sqrt{\varepsilon(M_r^2-1)}} - \arctan\frac{1}{\sqrt{M_r^2-1}} = \beta_s \tag{2.6}$$

(curve 2 in Fig. 1a).

It follows from (2.6) that the value of M_r decreases, as $|\beta_s|$ increases, and $M_r = 1$ for

$$|\beta_b| = \frac{\pi}{2}\frac{1-\sqrt{\varepsilon}}{\sqrt{\varepsilon}} \tag{2.7}$$

($\beta_b = -130.45°$, point b in Fig. 1a).

Thus, the domains of Mach numbers for which the system S_2 can exist are bounded by the functions $M_*(\beta_s)$ for $\beta_s > 0$ and $M_r(\beta_s)$ for $\beta_s < 0$; for $\beta_s > \beta_a$ and $\beta_s < \beta_b$, such a system does not exist for any Mach numbers.

In the domain of existence of S_2, the geometrical constraint is

$$\beta_s = \omega(M) - \omega(M_1(J_1)) + \beta_2(M_1(J_1), J_2), \tag{2.8}$$

and Eq. (2.3) solves the formulated problem.

3. As an example, a geometrically imposed system which is optimal for the static pressure ($f = p$) is considered. Figure 2 gives a qualitative pattern of the strength of the system $J_s = J_1 J_2$ as a function of the shock strength J_2 with a geometrical constraint $\beta_s = 0$ (Figs. 2a–2e correspond to M = 1.05, 1.2, 1.35, 1.6, 2.2, and 2.6). This pattern is constructed by numerical calculation of J_s using (2.8) to find J_1. It is evident that for small values of M (Fig. 2a), the function $J_s(J_2)$ is smaller than unity for any J_2 (overexpansion of the flow [5]) and has a minimum.

As M increases (Fig. 2b), the minimum is shifted to the coordinate origin, and the strength of the system increases with an increase in J_2. This gives rise to the range of J_2 values in which $J_s > 1$ (the underexpansion region [5]). When $M = M_{F_1}$ ($M_{F_1} = 1.245$), the value $J_2 = 1$ corresponds to the minimum. In the range $M \in [M_{F_1}, M_\Delta]$, where ($M_\Delta = 1.478$) (Fig. 2c), the function $J_s(J_2)$ is monotonically increasing. When $M = M_\Delta$ (the point Δ in Fig. 1b), the derivative $\partial J_s/\partial J_2$ vanishes at the point $J_2 = J_\Delta$, and when $M > M_\Delta$ (Fig. 2d), the static pressure behind the shock has a maximum for $J_2 < J_\Delta$ and a minimum for $J_2 > J_\Delta$. As M increases further (Fig. 2e), the strength of the system decreases again. As for small values

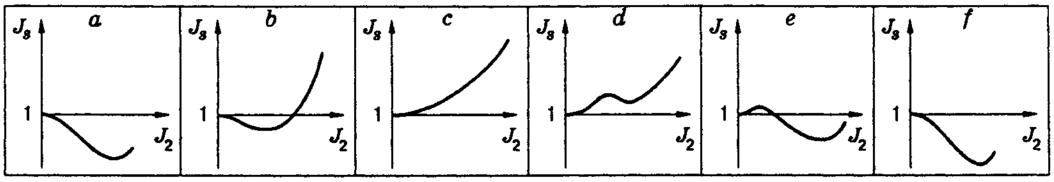

Fig. 2

of M, an overexpansion region appears in which $J_s < 1$, and the value of J_2 corresponding to the maximum tends to unity as M increases. When $\mathrm{M} > \mathrm{M}_{F_2}$ ($\mathrm{M}_{F_2} = 2.539$), the function $J_s < 1$ for any values of J_2, and it has a single extremum, namely, a minimum (Fig. 2f). A further increase in M does not change qualitatively the behavior of the static pressure behind the system. A similar pattern is observed when $\beta_s \neq 0$.

Thus, depending on the parameters M and β_s, four regions (see Fig. 1) in which the system strength varies differently can be distinguished. The function $J_s(J_2)$ is monotonic in region 2, and it has two extrema in region 3 and a single minimum in regions 1 and 4.

The boundaries of the regions and the strengths of optimal waves are found by solving Eq. (2.3) under constraint (2.8). For the static pressure, the relation (2.3) reduces to the following cubic equation in M_1:

$$\sum_{n=0}^{3} A_n (\mathrm{M}_1^2)^n = 0. \tag{3.1}$$

Here

$$A_3 = J_2^2(1+\varepsilon)^2 - 4\varepsilon(J_2+\varepsilon)^2;$$
$$A_2 = 4\varepsilon(1-\varepsilon)(J_2+\varepsilon)(J_2^2-1) - 2(1-\varepsilon^2)J_2^2(J_2-1) - 4(1-2\varepsilon)(J_2+\varepsilon)^2;$$
$$A_1 = (1-\varepsilon)[4(1-2\varepsilon)(J_2^2-1)(J_2+\varepsilon) + 4(J_2+\varepsilon)^2 + (1-\varepsilon)J_2^2(J_2-1)^2];$$
$$A_0 = -4(1-\varepsilon)^2(J_2+\varepsilon)(J_2^2-1).$$

In the range $J_2 \in [1, \infty)$, the equation has three real roots, which are presented in Fig. 3. The smallest root (curve 3) has no physical meaning, because it corresponds to $\mathrm{M}_1 < 1$. The two other roots (curves 1 and 2) together with the geometrical constraint

$$\omega(\mathrm{M}) = \beta_s + \omega(\mathrm{M}_1(J_2)) - \beta_2(J_2) \tag{3.2}$$

and a given value of β_s make it possible to find a relationship between the optimal shock strength and the free-stream Mach number. A particular case of this relationship ($\beta_s = 0$) is given in Fig. 4.

The medium root of Eq. (3.1) (curve 1 in Fig. 3) corresponds to the minimum of the function $J_s(J_2)$ at small Mach numbers (curve 1 in Fig. 4). The largest root (curve 2 in Fig. 3) determines the extrema for $\mathrm{M} > \mathrm{M}_\Delta$ (curve 2 in Fig. 4). It is evident from the graphs that the maximum and minimum of the function $J_s(J_2)$ are in the ranges $J_2 \in [1, J_\Delta]$ and $J_2 \in [J_\Delta, \infty)$, respectively.

Curves 3 and 4 described by the functions M_{φ_1} and M_{φ_2} bound from above regions 1 and 3 in Fig. 1. They intersect the axis of ordinates at points F_i. The formulas

$$\mathrm{M}_{F_i} = \sqrt{\frac{2}{5-3\gamma}[(3-\gamma) \mp \sqrt{\gamma^2-1}]} \qquad (i = 1, 2) \tag{3.3}$$

follow from Eq. (3.1) for $\beta_s = 0$, $J_2 \to 1$, and $J_1 \to 1$ ($\mathrm{M}_1 \to \mathrm{M}$). Previously, Chernyi [6] obtained these formulas by solving the problem of interaction of small perturbations with a shock. Uskov [4] obtained the same formulas by analysis of overtaking shocks in the plane β, Λ.

When $\beta_s < 0$, the functions $\mathrm{M}(J_2)$ do not differ qualitatively from the curves of $\mathrm{M}(J_2)$ in Fig. 4 for $\beta_s = 0$. Hence, to find the functions M_{φ_i} for $\beta_s < 0$ (the left branches of curves 3 and 4 in Fig. 1), it is

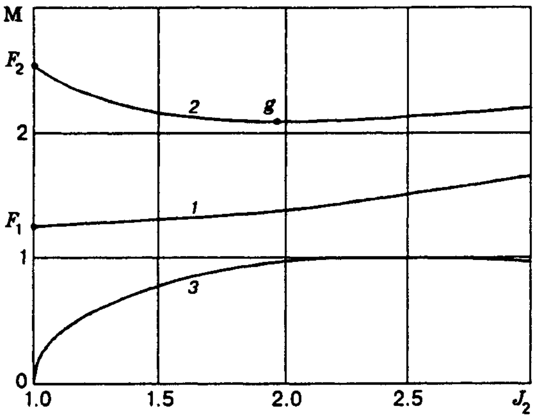

Fig. 3

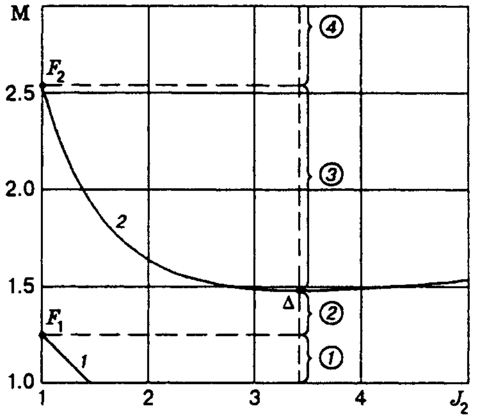

Fig. 4

necessary to set $J_2 = 1$ in (3.1). Then, relation (3.2) takes the form

$$\omega(\mathrm{M}) = \omega(\mathrm{M}_1) + \beta_s. \tag{3.4}$$

Here the values of M_1 are calculated from formulas (3.3), and they depend only on γ.

It is evident from (3.4) that as $|\beta_s|$ increases, the Mach numbers decrease monotonically and reach unity for

$$\beta_s = -\omega(\mathrm{M}_{F_i}) \tag{3.5}$$

(see points e and h in Fig. 1, $\beta_e = -4.7°$ and $\beta_h = -40.04°$). Thus, there are no regions 1 and 3 for $\beta_s < \beta_e$ and $\beta_s < \beta_h$, respectively.

For $\beta_s > 0$, a system exists if $\mathrm{M} > \mathrm{M}_*$ (2.4). The possible strengths J_2 in such a system are in the range $[J_*, \infty)$, where J_* is the shock strength calculated from M_* [formulas (2.4)]. Thus, for $\beta_s > 0$, the curves similar to those in Fig. 4 for $\beta_s = 0$ differ in that their initial points correspond to $J_2 = J_*$. The functions $\mathrm{M}_{\varphi_i}(\beta_s)$ describing the right branches of curves 3 and 4 in Fig. 1 are found by formula (3.1) from the condition $J_2 = J_*$. Since $\beta_2(J_2) = \beta_s$ for $J_2 = J_*$, it follows from (3.2) that $J_1 = 1$, $\mathrm{M}_1 = \mathrm{M}$, and the Mach numbers M_{φ_i} are the medium and largest roots of Eq. (3.2) (curves 1 and 2 in Fig. 3).

The shock strength J_c corresponding to the point c of intersection of curves 1 and 3 in Fig. 1b is determined as the third root of the cubic equation

$$4\varepsilon J_c^3 + 3(1-\varepsilon)^2 J_c^2 - (5\varepsilon^2 - 2\varepsilon + 5)J_c + 1 - 3\varepsilon - \varepsilon^2 - \varepsilon^3 = 0, \tag{3.6}$$

which is obtained by simultaneous solution of Eqs. (2.4) and (3.1). Here $J_c = 1.466$, $\mathrm{M}_c = 1.305$, and $\beta_c = 6.46°$. When $\beta_s > \beta_c$, region 1, which exists only for angles $\beta_s \in [\beta_e, \beta_c]$, disappears.

The function $\mathrm{M}_{\varphi_2}(\beta_s)$ has a minimum at point g (Fig. 1a), which corresponds to the shock strength J_g. The value of J_g is found by solving the equation

$$\sum_{n=0}^{3} B_n (\mathrm{M}_{\varphi_2}^2)^n = 0, \tag{3.7}$$

where $B_3 = 2J_g(1+\varepsilon)^2 - 8\varepsilon(J_g+\varepsilon)$, $B_2 = 4\varepsilon(1-\varepsilon)y - 2(1-\varepsilon^2)J_g(3J_g-2) - 8(1-2\varepsilon)(J_g+\varepsilon)$, $B_1 = 2(1-\varepsilon)[2(1-2\varepsilon)y + 4(J_g+\varepsilon) + (1-\varepsilon)J_g(J_g-1)(2J_g-1)]$, $B_0 = -4(1-\varepsilon)^2 y$, $y = (J_g^2 - 1) + 2J_g(J_g+\varepsilon)$, and M_{φ_2} is calculated from formula (3.1) ($J_g = 1.989$, and $\mathrm{M}_g = 2.089$, and $\beta_g = 12.7°$).

The Mach number M_Δ separating regions 2 and 3 for a given value of β_s corresponds to a minimum (point Δ in Fig. 4) of the implicit function $\mathrm{M}(J_2, \beta_s)$ given by formula (3.2) under the constraint (3.1). To determine the minimum Eq. (3.2) can be rewritten as

$$\nu(\mathrm{M}) \equiv \omega(\mathrm{M}) - \beta_s = \omega(\mathrm{M}_1(J_2)) - \beta_2(J_2). \tag{3.8}$$

The left-hand side of (3.8) is a monotonic function of M, and the right-hand side depends only on J_2. Hence, if the function $\nu(\mathrm{M})$ has a minimum for some $J_2 = J_\Delta$, $\mathrm{M}(J_2)$ reaches a minimum for the same J_2. The strength J_Δ does not depend on β_s.

Testing (3.8) for an extremum leads to the following equation for J_Δ:

$$J_\Delta(1+\varepsilon)\sum_{n=0}^{2}(n+1)A_{n+1}(J_\Delta)(\mathrm{M}_1^2)^n + 2\mu_1\sum_{n=0}^{3} B_n(J_\Delta)(\mathrm{M}_1^2)^n = 0. \tag{3.9}$$

Here $A_{n+1}(J_\Delta)$ and $B_n(J_\Delta)$ are coefficients of Eqs. (3.1) and (3.7), respectively; $\mathrm{M}_1 = \mathrm{M}_1(J_\Delta)$ and is calculated as the largest root of the cubic equation (3.1) ($J_\Delta = 3.434$ and $\mathrm{M}_{1\Delta} = 2.282$).

Thus, for any β_s, the function $\mathrm{M}_\Delta(\beta_s)$ is described by the relation

$$\omega(\mathrm{M}_\Delta) = \beta_s + \nu(\gamma), \tag{3.10}$$

and it is a monotonic function of β_s (curve 5 in Fig. 1). Curves 4 and 5 intersect at point d, whose coordinates are calculated from the conditions $\mathrm{M}_\Delta = \mathrm{M}_{1\Delta}$ and $\beta_d = \beta_2(J_\Delta, \mathrm{M}_\Delta)$, where $\beta_d = 22.56°$. Curve 5 intersects the abscissa at point q. The angle β_q corresponding to this point is found from (3.10) subject to the condition $\mathrm{M}_\Delta = 1$ $[\omega(\mathrm{M}_\Delta) = 0]$. In this case $\beta_q = -11.27°$.

Thus, region 3 exists only in the range of angles $[\beta_q, \beta_d]$.

4. For fixed values of β_s and M, the shock strength J_2 can vary in the range $[J_\sigma, J_\vartheta]$. The left bound J_σ is a function only of the angle β_s: the strength $J_\sigma \equiv 1$ for $\beta_s \leqslant 0$, which corresponds to rotation of the expansion flow through the angle β_s, and $J_\sigma = J_*(\beta_s)$ for $\beta_s > 0$ [see (2.4)]. The right bound J_ϑ is determined from Eq. (2.8) subject to the condition $J_\vartheta = J_*(\mathrm{M}_1)$.

The extrema of the function $J_s(J_2)$ are found within the interval (J_σ, J_ϑ), and the boundary points $J_2 = J_\sigma$ and $J_2 = J_\vartheta$ are local extrema. In this case, as can be seen in Fig. 2, $J_\sigma = 1$ corresponds to the local maximum of the function $J_s(J_2)$ in the ranges $\mathrm{M} \in [1, \mathrm{M}_{F_1}]$ and $\mathrm{M} \in [\mathrm{M}_{F_2}, \infty)$. For $\mathrm{M} \in [\mathrm{M}_{F_1}, \mathrm{M}_{F_2}]$ the value of J_σ determines the local minimum of the system strength.

The global maximum of the static pressure behind the system S_2 for a given value of β_s is reached when $J_2 = J_\vartheta$ and $\mathrm{M} = \mathrm{M}_w$. The Lagrange method is used to determine M_w. The Lagrangian

$$L_m = J_\vartheta\left(\mu\frac{J_\vartheta + \varepsilon}{J_\vartheta(1+\varepsilon J_\vartheta)}\right)^{(1+\varepsilon)/2\varepsilon} + \lambda\left[\omega(\mathrm{M}_1) - \omega(\mathrm{M}) - \beta_*(J_\vartheta) + \beta_s\right] \tag{4.1}$$

depends on three variables: J_ϑ, M, and λ. The value of M_1 is found on the condition $\mathrm{M}_2 = 1$ from the equation

$$\mu_1 = J_\vartheta(1+\varepsilon J_\vartheta)/(J_\vartheta + \varepsilon), \tag{4.2}$$

and the angle $\beta_*(J_\vartheta)$ is found from (2.4).

Differentiating (4.1) with respect to J_ϑ, M, and λ with allowance for (4.2) and eliminating the Lagrange multiplier λ, we obtain the following equations for the Mach number M_w:

$$\beta_s = \beta_*(J_\vartheta) + \omega(\mathrm{M}_w) - \omega(\mathrm{M}_1(J_\vartheta)); \tag{4.3}$$

$$\mathrm{M}_w^2 = (x^2 \pm \sqrt{x^4 - 4x^2})/2. \tag{4.4}$$

Here

$$x = \frac{1}{\gamma}\,\frac{(J_\vartheta - 1)^2}{J_\vartheta(J_\vartheta + \varepsilon)(1+\varepsilon J_\vartheta)}[\zeta - \xi]^{-1}; \qquad \zeta = \frac{\sqrt{\mathrm{M}_1^2 - 1}}{\mathrm{M}_1^2\mu_1}\,\frac{J_\vartheta^2 + 2\varepsilon J_\vartheta + 1}{(J_\vartheta + \varepsilon)^2};$$

$$\xi = \sqrt{\frac{J_\vartheta - 1}{1+\varepsilon J_\vartheta}}\,\frac{2J_\vartheta + 1}{J_\vartheta(J_\vartheta^2 + J_\vartheta - 1 + \varepsilon)}.$$

It is evident from (4.3) and (4.4) that the function $\mathrm{M}_w(\beta_s)$ (curve 6 in Fig. 1) is parametric. The quantity J_ϑ serves as a parameter.

The radicand in (4.4) vanishes if $x = 2$; in this case $\mathrm{M}_w = \sqrt{2}$, $J_\vartheta = J_z$ ($J_z = 3.882$), and $\beta_s = \beta_z$ ($\beta_z = 4.51°$, see point z in Fig. 1b). In the range $J_\vartheta \in [J_z, \infty)$, the radicand is larger than zero; the plus sign of the root corresponds to $\mathrm{M}_w \in [\sqrt{2}, \infty)$, and the minus sign corresponds to $\mathrm{M}_w \in (1, \sqrt{2}]$. The angle $\beta_s \to \beta_b$ for $\mathrm{M}_w \to 1$ [see (2.7)] and $\beta_s \to \beta_a$ for $\mathrm{M}_w \to \infty$ [see (2.5)] (Fig. 1b). Thus, the global extremum of the static pressure is reached for any β_s from the domain of existence of the system S_2.

5. The geometrically imposed systems $S_2^{(f)}$ can be optimal not only for the static pressure, but also for the temperature, density, and velocity head. The boundaries of the nonmonotone range and extrema of functions are found from Eq. (2.3) subject to condition (3.2).

Equation (2.3) has the simplest form for the temperature $[I^{(f)} = I^{(T)}]$:

$$\mathrm{M}_1^2 = (J_2 + 1)\frac{J_2(J_2 + \varepsilon) + (1 + \varepsilon J_2)}{J_2(J_2 + 1)(1 + \varepsilon) + (1 + \varepsilon J_2)}. \tag{5.1}$$

For given M and β_s, this makes it possible to consider (2.8) as an equation for a single unknown J_2.

The calculations performed show that for small Mach numbers ($\mathrm{M} \in [1, \mathrm{M}_t]$), the temperature has a minimum for $J_2 = J_t$, which is determined from Eq. (3.2) subject to condition (5.1). For $\mathrm{M} > \mathrm{M}_t$, the function $I^{(T)}(J_2)$ is monotonic.

For $\beta_s > 0$, the function $\mathrm{M}_t(\beta_s)$ (curve 7 in Fig. 1b) is found from (3.2) subject to the condition that $\mathrm{M} = \mathrm{M}_1(J_2)$ [see (5.1)]. For $\beta_s < 0$, one should set $J_2 = 1$ in (3.2); in this case, $\mathrm{M}_1 = 2/\sqrt{3}$, as can be seen from (5.1).

The nonmonotone range of $I^{(T)}(J_2)$ exists for the range of angles $[\beta_u, \beta_v]$. The angle β_u (point u in Fig. 1b) is determined from (3.2) at $\mathrm{M} = 1$, $J_2 = 1$, and $\mathrm{M}_1 = 2/\sqrt{3}$ in the form

$$\beta_u = \frac{1}{\sqrt{3}}\arctan\sqrt{\frac{\varepsilon}{3}} - \frac{\pi}{6}$$

($\beta_u = -2.49°$). The strength J_v calculated by the formula

$$J_v = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{27 - 4(1 + \varepsilon)^3}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{27 - 4(1 + \varepsilon)^3}{108}}} \tag{5.2}$$

corresponds to the coordinates of point v (Fig. 1b). Formula (5.2) is obtained by simultaneous solution of Eqs. (2.4) and (5.1).

The values of M_v [formula (5.1)] and β_v [formula (3.2)] can be determined from the known value of J_v ($\mathrm{M}_v = 1{,}257$ and $\beta_v = 5.16°$).

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